Sci Am. (212) 754-0550 415 Madison Ave NY 10017

MATHEMATICAL GAMES

Fantastic patterns traced by programmed "worms"

by Martin Gardner

Fashionable methods of teaching mathematics to children come and go. (At the moment the modern-math fad is mostly going.) One of these days mathematics teachers will discover what John Dewey tried to tell them 75 years ago: Children learn best by doing something they enjoy. With this in mind, Seymour A. Papert, a former assistant to Jean Piaget who is now working in the Artificial Intelligence Laboratory of the Massachusetts Institute of Technology, has designed a variety of animal robots that can be controlled by a desk computer. One of them is a "turtle" with a pen on its underside. Suitably programmed, the turtle draws geometric figures by crawling across large sheets of paper on the floor.

Defining geometric figures as paths generated by a moving point is an ancient idea. Take a square. Instead of calling it a four-sided polygon with equal sides and angles, call it the path traced by a worm crawling over a plane according to the following rule: Go straight for distance k, turn 90 degrees left and repeat until the path returns to its origin.

The idealized worm (moving point) can obviously be programmed to generate any pattern of lines. A challenging recreational task now presents itself. What kinds of program, with extremely simple rules, give interesting or beautiful patterns? A good way to simplify rules is to restrict the worm to paths along a regular lattice. This enables one to experiment with rules by drawing paths on square or isometric graph paper. Better still, if one has access to a computer with a display screen, one can write simple programs, then enjoy the spectacle of watching a path of light grow on the screen.

Frank C. Odds, a British biochemist, recently proposed a class of rules for generating patterns that he named spirolaterals. The worm crawls a distance of one unit, turns, crawls two units, turns, crawls three units and so on, traversing distances in counting order until the length of a path segment reaches a specified integer, n, when the procedure starts over. The turning angle is always the same, but the direction of turn may be right or left according to the worm's program. The number n, which is the number of segments in the counting series and also the number of turns before the series repeats, is called the order of the spirolateral.

Two examples will make this clear. If the order is 1, the angle is 90 degrees and all turns are in the same direction, the spirolateral is a square. If the order is 7, the angle is 90 and all turns are the same, the spirolateral has a pleasing closed pattern. The top illustration on page 118 shows all the right-angle spirolaterals of orders 1 through 8, when all turns are the same. It is easy to see why Odds chose the name spirolateral: "lateral" for flat surface and "spiro" for the square spirals that generate the figures.

Note that spirolaterals of orders 4 and 8 do not close (return to the origin). As Odds puts it, they meander jerkily to infinity. Indeed, it is not hard to prove that there is no closure for orders in the series 4, 8, 12, 16..., that two repetitions of the square spiral will close the figure (producing twofold symmetry) if the order is in the series 2, 6, 10, 14..., and that all other orders close after four repetitions and have fourfold symmetry.

There is no reason why all turns must be the same way, but when they vary, the situation becomes complicated enough to call for a compact notation. Odds suggests writing the angle as a subscript of the order number and using a superscript on the left or the right to show which turns are left and which right. For example, ${}^{1,7}9_{90}$ defines the spirolateral generated by nine turns of 90 degrees, with left turns preceding segments 1 and 7 and (by implication) all the other segments turning right. The same spirolateral could be defined by putting a superscript on the right to indicate right turns, $9_{90}{}^{2,3,4,5,6,8,9}$, but clearly the first notation is preferable.

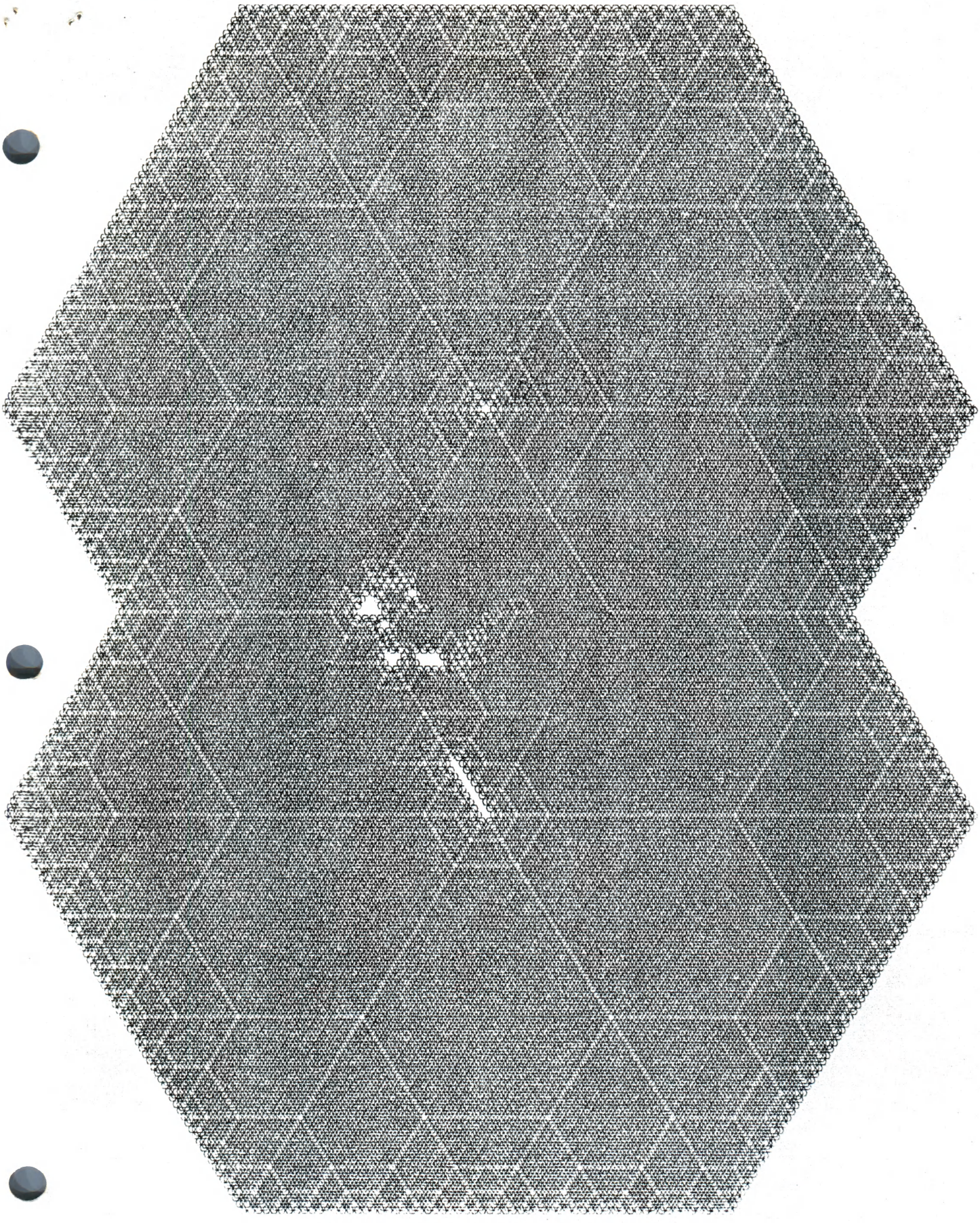
The middle illustration on page 118 shows two right-angle spirolaterals with mixed left and right turns. The bottom illustration shows spirolaterals with angles of 36, 45 and 60 degrees. Spirolaterals with 60-degree turns can be easily drawn on isometric paper. Those with 45-degree angles are not hard to construct with a ruler and a draftsman's triangle, but those with other angles take more doing. Any angle that is an exact divisor of 180 degrees will generate a spirolateral.

Not much is known about spirolaterals. Sometimes two different rules produce spirolaterals that are mirror images (for example ${}^{1,2,3}7_{90}$ and ${}^{5,6,7}7_{90}$), but no one knows how to predict this without drawing the figures. Nor is it known how to tell from looking at a spirolateral's formula, except in special cases, whether or not the figure will close and, if it does, how many repetitions are needed to close it.

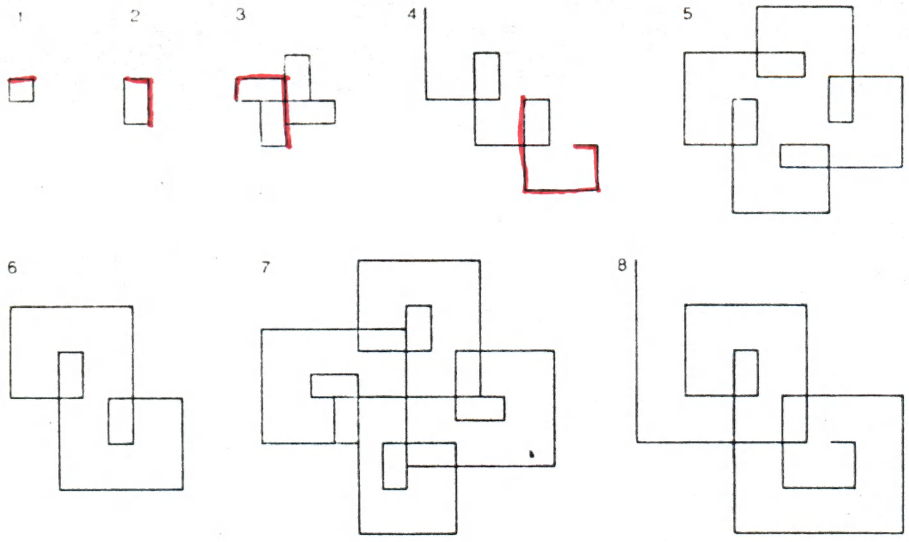
A few years ago John Horton Conway of the University of Cambridge suggested a new approach to worm paths. Instead of viewing the worm as an explorer, view it as an eater. Food is confined to the lines of an arbitrarily large grid. The worm hatches from an egg at one node and then starts to crawl along grid lines, eating as it goes. A fixed set of rules determines the worm's decision at each node. It is assumed that the worm never traverses a segment already eaten. No segment lengths need to be specified, only the direction of turn, the direction depending solely on the state (eaten or uneaten) of all segments meeting at that node.

"Paterson's Worm," by Michael Beeler, a memorandum issued in June of this year by M.I.T.'s Artificial Intelligence Laboratory, deals entirely with such worm tracks. What follows, either directly quoted or paraphrased, is taken from the memorandum with the permission of Beeler and Marvin L. Minsky, who heads the laboratory. (Readers can obtain the document by sending 80 cents to Room 912, 545 Main Street, Cambridge, Mass. 02139.)

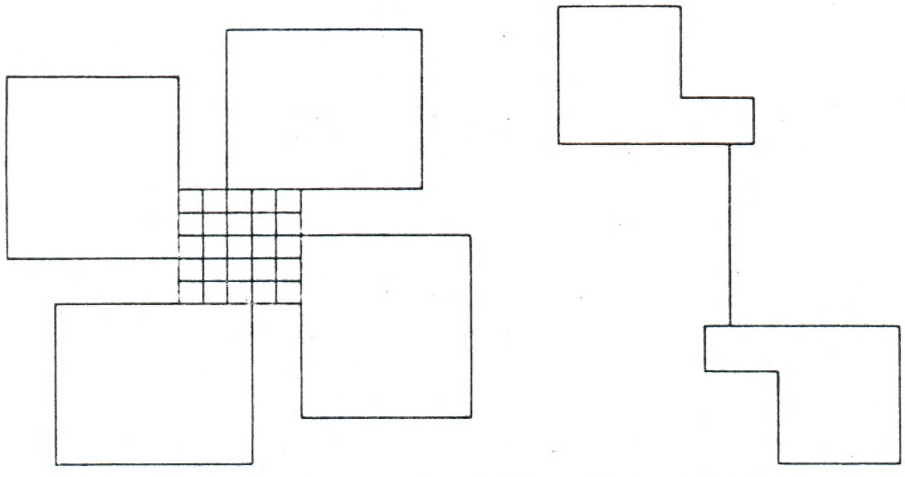
"Certain prehistoric worms fed on sediment in the mud at the bottom of ponds," Beeler's memorandum begins. "For efficiency, they would not retrace paths which had already been traveled, since little food was left there. Yet food probably occurred in patches, so it was desirable to stay near previous trails. Worms had innate 'rules' regarding how close to 'eaten paths' to stay, how far to



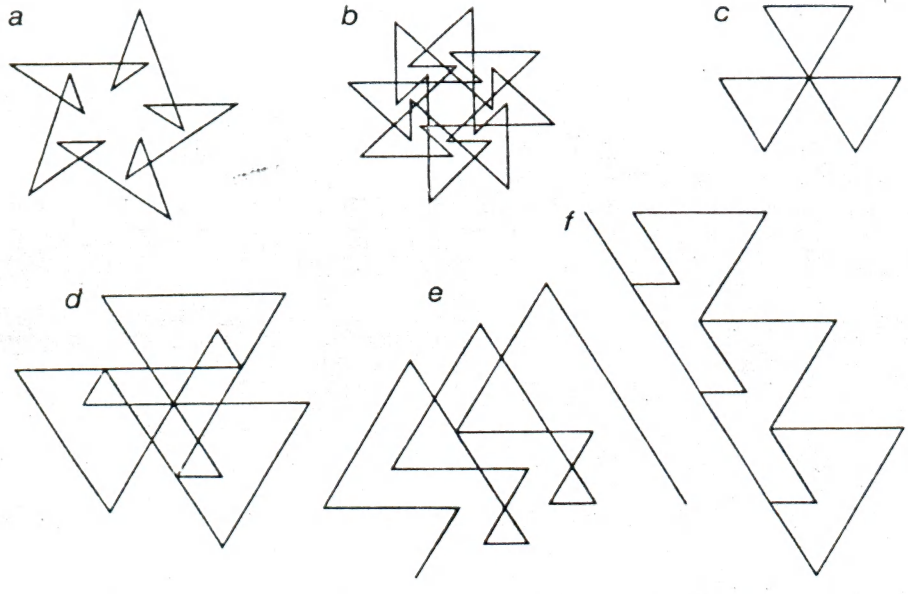
Longest known finite path generated by a simple isometric worm



Spirolaterals of orders 1 through 8 with all turns in same direction



Ninety-degree spirolaterals with mixed turns: 9_{90}^{6} (left) *and* $8_{90}^{1,4,8}$ (right)



Spirolaterals (a) 3_{36}, (b) 13_{45}, (c) 2_{60}, (d) $1,25_{60}$, (e) 35_{60} *and* (f) 25_{60}

go before turning around, how sharp a turn to make, etc. These rules varied from species to species, and paleontologists can trace the development of species and determine the similarity of different species by comparing fossil records of worm tracks. (See *Science* magazine, 21 November 1969, for further details and a discussion of computer simulation of natural worm tracks.)

"Early in 1971, Michael Paterson [a computer scientist at the University of Warwick] mentioned to me a mathematical idealization of the prehistoric worm. He and John Conway had been interested in a worm constrained to eat food only along the grid lines of graph paper. . . .

"If a worm, arriving at a node with no segments eaten (except of course the one it just ate), should find in its rules, 'For this distribution, go straight,' then the worm will go straight forever. Since this is neither interesting to us, nor very useful to a real worm, which would quickly reach the edge of its food patch, we discard it. We require that, upon discovering a virgin node, all sets of rules must say to turn. To avoid mirror-image duplication, we require that the turn be to the worm's right (clockwise as seen from above)."

Consider what Beeler calls a "simple quadrille worm": a worm crawling on a square grid and turning right at each node. What happens after it traces a square? It cannot turn right again because this would take it along an eaten segment. It has only two choices. If it is programmed to turn left if, and only if, it cannot turn right, it will trace two squares [*see "a" in top illustration on opposite page*]. No uneaten segments are available, and so the worm dies. If it is programmed to go straight when it cannot go right, and left when it cannot go straight or right, it will trace five squares before it expires at its origin ["*b*"]. These two fossil tracks exhaust the variety of species of simple quadrille worms.

To avoid a worm's early demise, Conway proposed that quadrille worms have the ability to "look ahead" and see the distribution of eaten and uneaten segments at each adjacent node. For example, a worm could be programmed to turn right if, and only if, it senses that this will take it to a node where four uneaten segments meet. Otherwise it goes straight. The result is a simple square spiral. If the rule is to turn left whenever a right turn leads to a node with an eaten segment, the path is a more interesting spiral [*see middle illustration*

at right]. It is easy, of course, to produce more elaborate paths by complicating the rules.

The rules may allow anything. What happens when rules allow look-ahead worms to hop? What happens when barriers are suitably placed or when the field is bounded on all sides? What happens when two or more worms of the same or different species interact? What happens when a newly hatched worm crawls a short distance along a defined path (such as a straight line of three units) before its repetitive behavior starts? How about two armies of worms crawling toward each other, each army obeying a different program? Are there possibilities here for competitive games? Are there interesting paths or patterns in three dimensions and higher dimensions?

Beeler avoids such difficult questions by confining his attention to what I shall call "simple isometric worms": worms that feed along an isometric grid of unit equilateral triangles. They are simple worms because they do not look ahead. With the isometric grid, however, six segments (not four) meet at each node. It seems to be a trivial difference, but as Beeler makes clear the possibilities for variant rules allow the definition of no fewer than 1,296 species.

All simple worms, quadrille or isometric, obey three general rules:

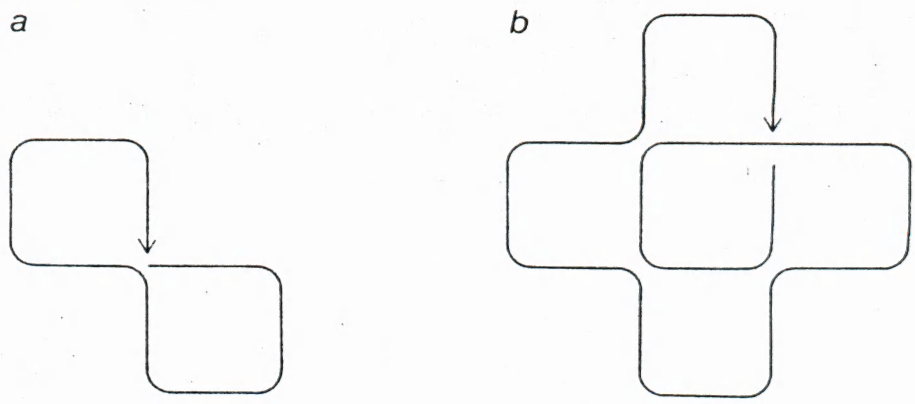
1. If no segments have been eaten at a node (except the segment just traveled), the worm turns right.
2. If all segments at the node have been eaten, the worm dies.
3. If only one segment at the node is uneaten, the worm takes it.

As we have seen, a simple quadrille worm, following the above rules, encounters only one "field" in which it must choose. Since it has only two choices, we can define only two species. The isometric grid, in contrast, offers a simple worm four major fields (one consisting of four subfields) in which decisions must be made. It is the behavior of these simple isometric worms that Paterson became interested in, and Beeler's memorandum also is primarily concerned with their behavior.

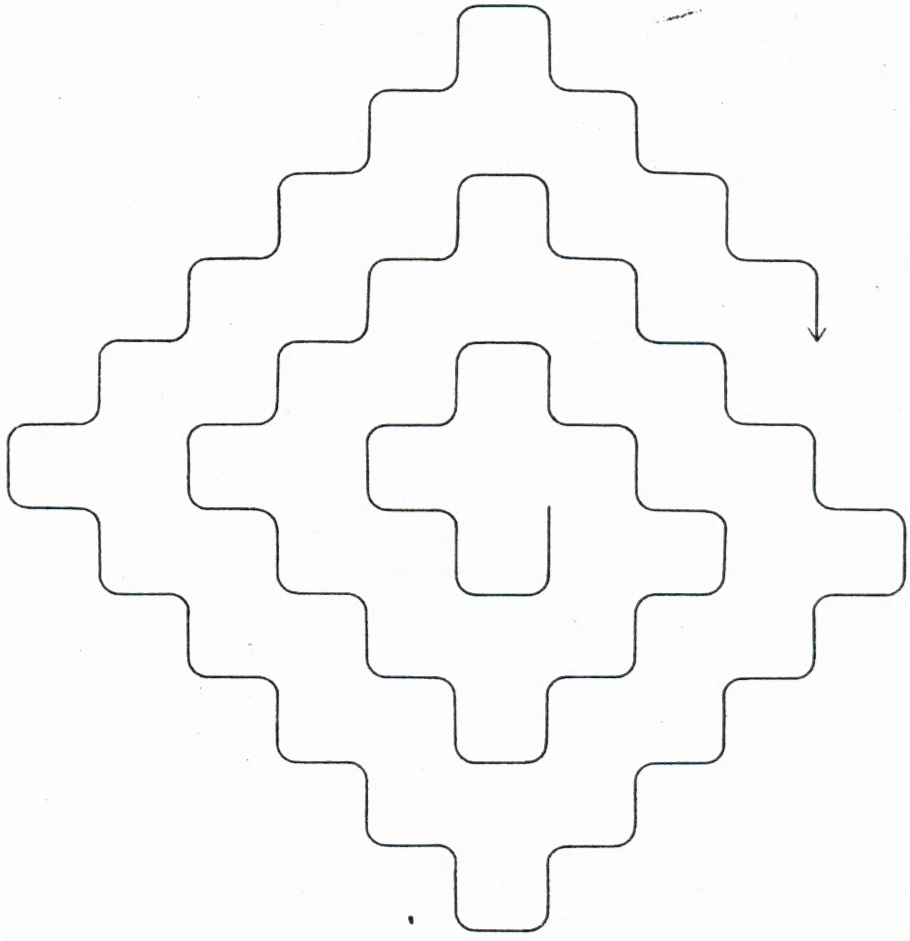
The four major fields, and all their choices, are shown in the illustration on the next page. The black lines indicate uneaten segments, the light colored lines are eaten segments of the worm's path and the colored arrows show how the worm approaches and leaves a node.

The four fields are:

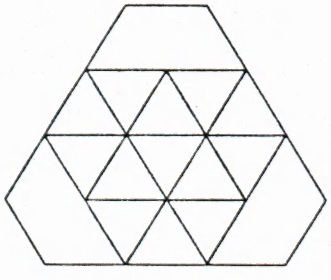
1. The worm approaches a node with two eaten segments other than the one it has just traversed. Its right turn can be



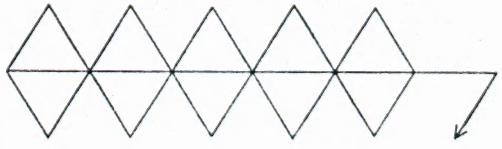
Fossil tracks of the only two species of simple quadrille worms



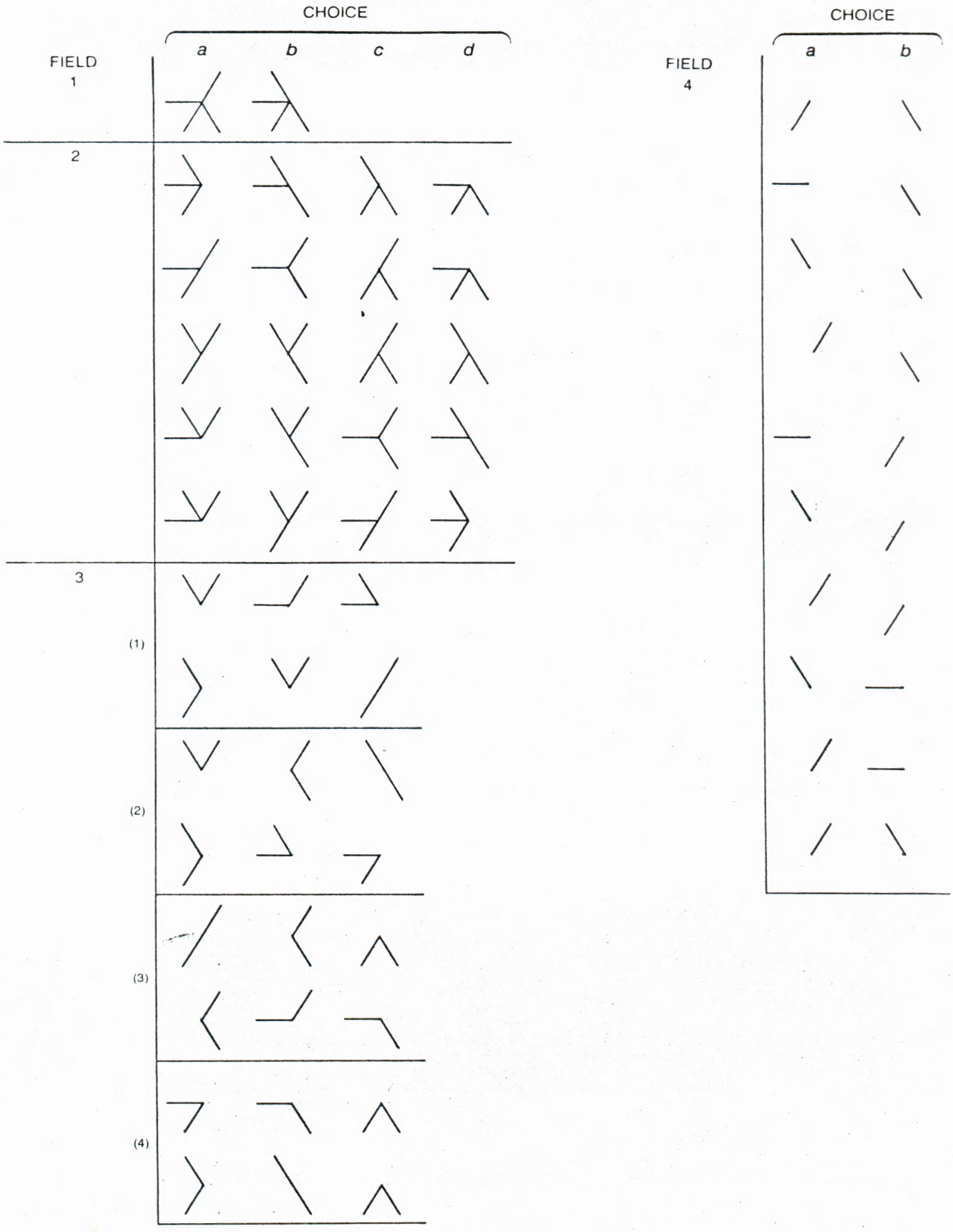
Infinite path of a "look ahead" quadrille worm that turns left when it cannot turn right



Path traced by 14 simple isometric worms



Zipper path of 54 sharp worms



Fields of choice for simple isometric worms

either "gentle" (120 degrees) or "sharp" (60 degrees). Number of choices: two.

2. The worm meets one eaten segment as it returns for the first time to its origin. As the chart shows, it can approach this node along any of five different segments. For each approach there is a choice of leaving by one of four uneaten segments. Number of choices: four.

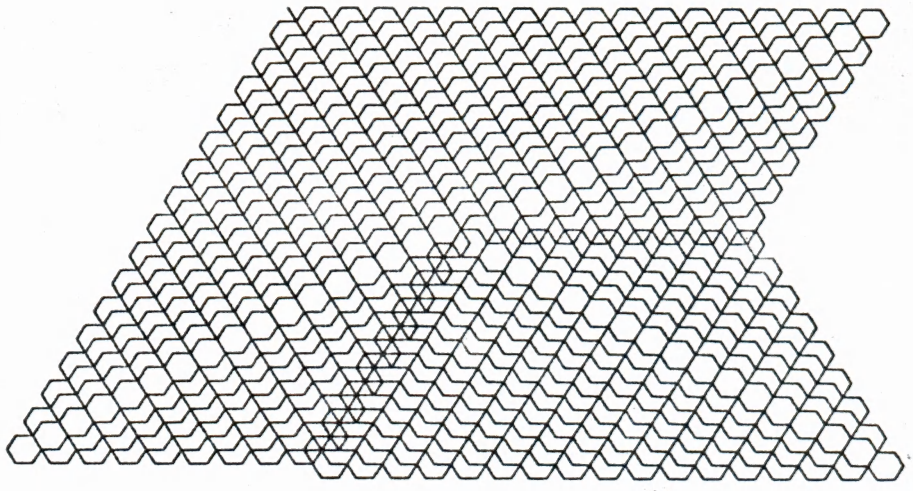
3. The worm meets two eaten segments as it returns to a point along its path. The node will be the vertex of either a sharp turn or a gentle one. In either case the approach can be made in four ways. For each way the worm can leave by one of three uneaten segments. Number of choices: $3 \times 3 \times 3 \times 3 = 81$.

4. The worm meets three eaten segments as it returns for the second time to its origin. This can happen in 10 ways, but each requires only a choice between two ways of leaving the node. Number of choices: two.

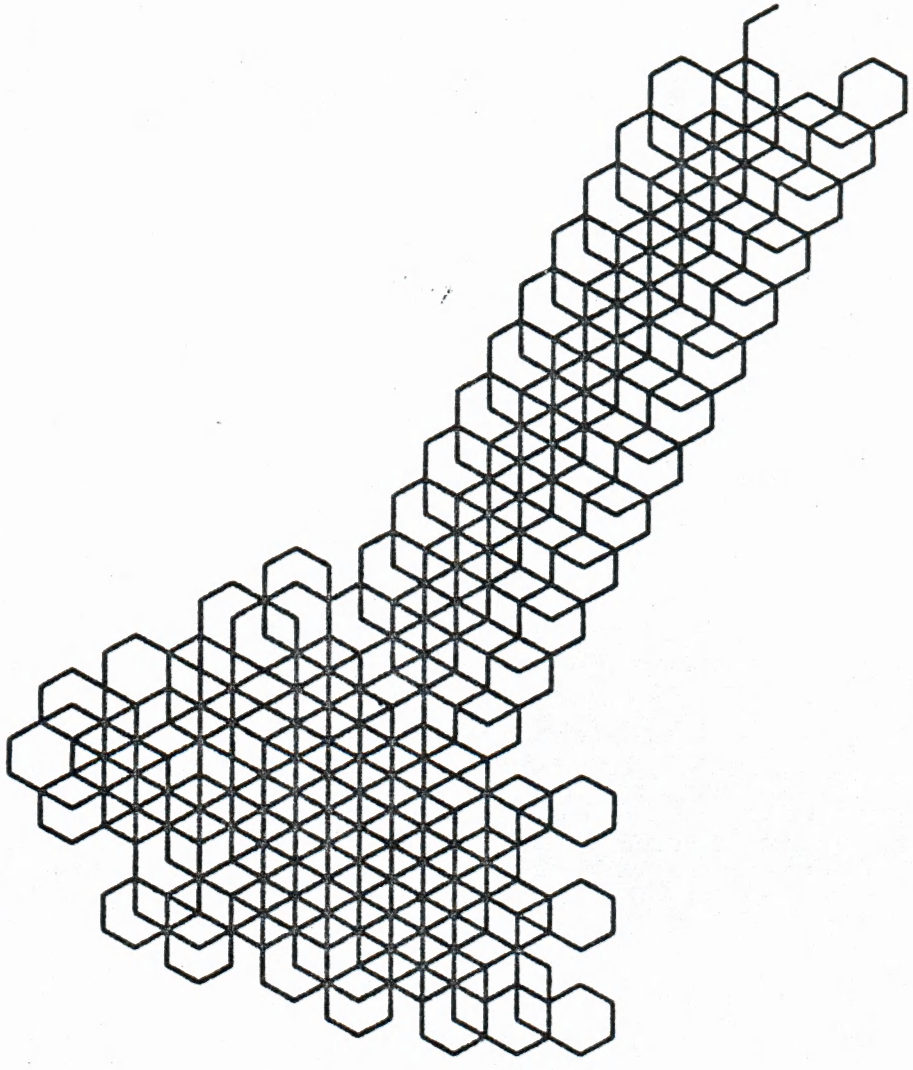
Meeting four or five eaten segments does not offer the worm a choice. In the first case it must take the only remaining uneaten segment. In the second case it has no segment to take, and so it dies. Thus there are $2 \times 4 \times 81 \times 2 = 1{,}296$ sets of rules, each defining a distinct species of simple isometric worm.

Beeler uses the pattern shown in the bottom illustration at the left on page 119 to explain how the rules work. The notation adopted here (Beeler uses a more compact notation based on octal and binary digits) identifies this as a fossil path generated by a $1_a2_b3_{acac}4_b$ worm. The formula tells us that when the worm faces a choice in Field 1, it follows Rule *a*. (It makes a gentle, not a sharp, turn.) For Field 2 it chooses *b*. The four subscripts after 3 refer to the four subfields of Field 3, where choices *a*, *c*, *a*, *c* are in the worm's feeding program. Finally, for Field 4 the worm selects *b*. As Beeler suggests, the reader might pause at this point to see if he can trace the path on a sheet of isometric paper, following the seven rules given by the formula. (It is necessary at each node to rotate the paper until the pattern at that node corresponds to the proper diagram on the chart.)

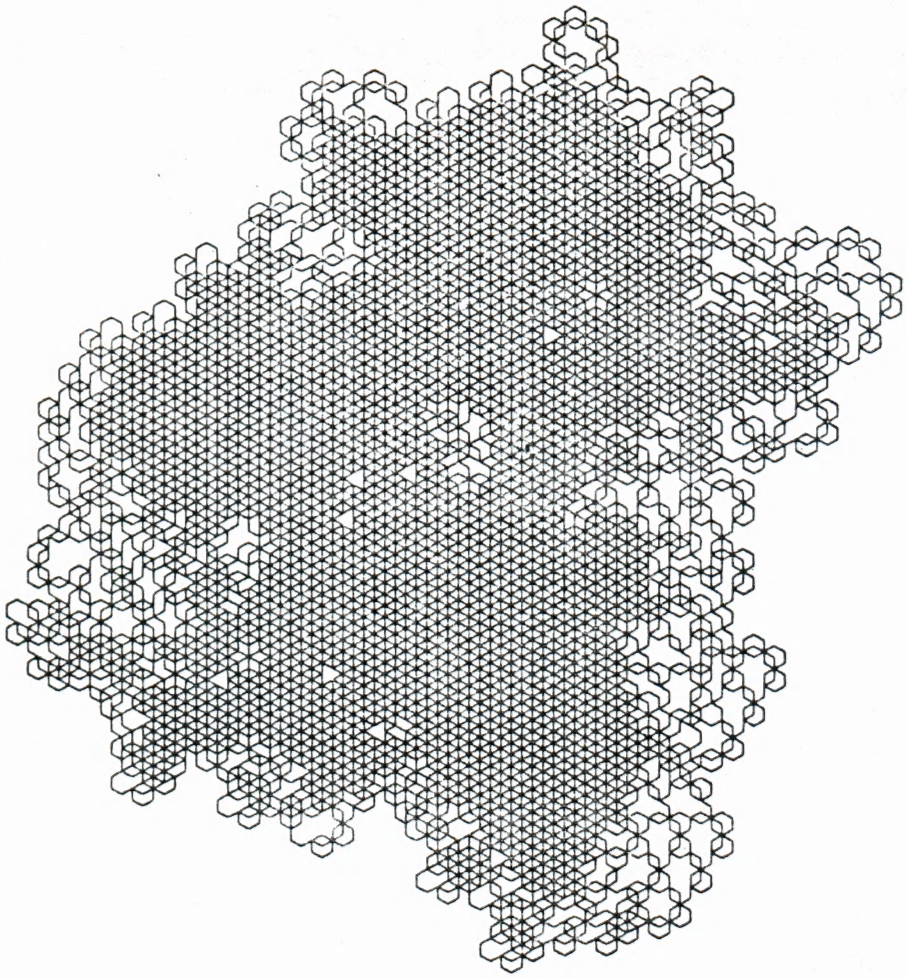
The path just described is generated by 14 worms. Some trace it exactly the same way. For example, the choice presented by the first subfield of Field 3 never arises, therefore it does not matter which of the three choices, *a*, *b* or *c*, is in the program. We indicate this in the formula by putting the three choices inside parentheses: $1_a2_b3_{(abc)cac}4_b$. The formula now describes three different



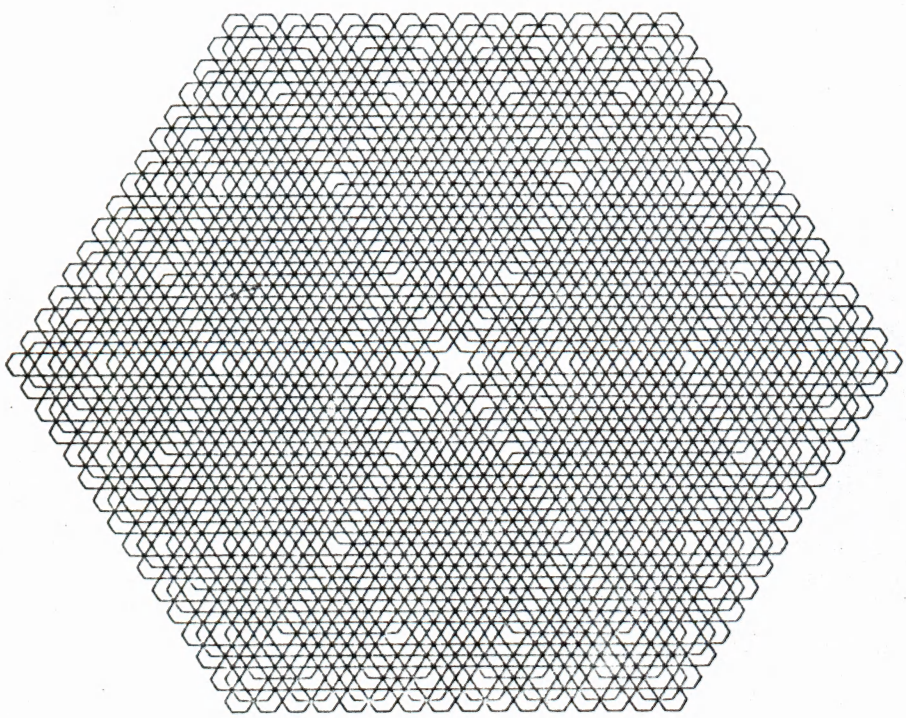
Three-pointed spiral by a $1_a2_c3_{acba}4_a$ *worm*



Shoot grower by a $1_a2_a3_{caaa}4_b$ *worm*



Cloud path generated by a gentle worm $1_a2_{abaa}3_c4$



The superdoily ($1_a2_d3_{cbaa}4_b$)

worms. Each draws the same path the same way. Other worms draw the same path different ways.

Beeler's computer program explored the behavior of all 1,296 species of simple isometric worm. Results show that 209 species generate paths that are unique, in the sense that no other species generates them. Forty-six paths are each characteristic of two species, and 44 paths belong to more than two. Thus there are 299 distinct paths all together.

The simplest of the 299 paths is the radioactivity symbol [*see "c" in bottom illustration on page 118*]. It has the smallest length (nine units) and also the fewest number of nodes (seven). It is the path traced by the largest number of worms, no less than 162 species. Using parentheses to show alternate choices, one formula defines all 162:

$$1_b2_{(ac)}3_{(abc)(abc)(abc)(abc)}4$$

The formula shows at once that it describes $2 \times 3 \times 3 \times 3 \times 3 = 162$ sets of rules. If the reader will test this formula on isometric paper, he will find that, regardless of the choice he makes inside each pair of parentheses, the radioactivity symbol will result.

Is there a longest path? No, because many paths never return to their origin a third time and are therefore infinite. A trivial infinite path of a type called the zipper is produced by 54 sharp worms with the formula $1_b2_d3_{(abc)(abc)b}4_{(ab)}$ [*see bottom illustration at right on page 119*].

Other worms spiral forever around their origin. The spiral shapes vary enormously: hexagons, diamonds, triangles, stars and asymmetric shapes such as the one shown in the top illustration on the preceding page. "Shoot growers," another class of infinite paths, start in a conventional way, and then the worm falls into a curious spiraling action that keeps repeating itself with regular displacements, creating a rod that shoots off to infinity [*see bottom illustration on preceding page*].

The longest of the known finite paths is shown in the illustration on page 117. The path is generated by a $1_a2_d3_{cbac}4_b$ worm and has 220,142 units. Note the crystalline regularity of the border and the lines that crisscross near the border, and how strikingly this symmetry contrasts with the dishevelment near the center. The point of origin, not identifiable, is at a spot just to the left of the center of the pattern.

One speaks of "known" finite paths because about a dozen worms have paths

so long that no one yet knows whether they are finite or infinite. The path generated by $1_b2_a3_{brau}4_b$, for instance, was tracked by Beeler to a length of 10 million units without revealing whether the worm dies or goes on gorging itself forever.

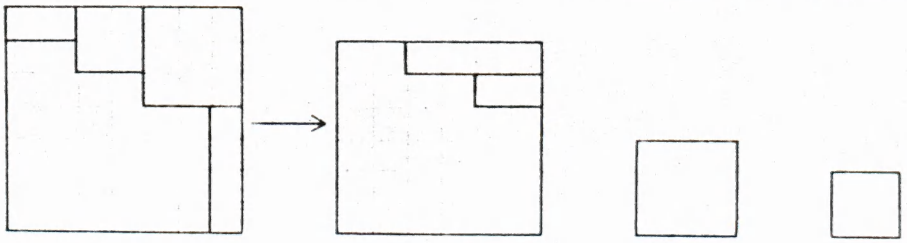
Some path patterns have outlines as unstructured as a cloud [*see top illustration on opposite page*]. Others, such as the "superdoily," display the rigid sixfold symmetry of a snow crystal [*see bottom illustration on opposite page*]. Note the six-pointed star in the center. Occasionally one such star or more appear as randomly situated white spots inside a gray plenum of closely packed unit triangles. Sometimes the stars are separated; sometimes they are partially merged to form binary or ternary systems. The situation is not unlike that of the physical world, in which low-order mathematical laws generate a rich assortment of both patterned and unpatterned objects, as well as objects in which order and disorder, symmetry and asymmetry are crazily mixed.

Beeler's work clearly has just scratched the surface of isometric-worm pathology. His program investigated only the simplest genus, with no attempt to complicate the genetic rules or to find out what happens when simple worms interact with barriers, boundaries or other worms of the same or different species.

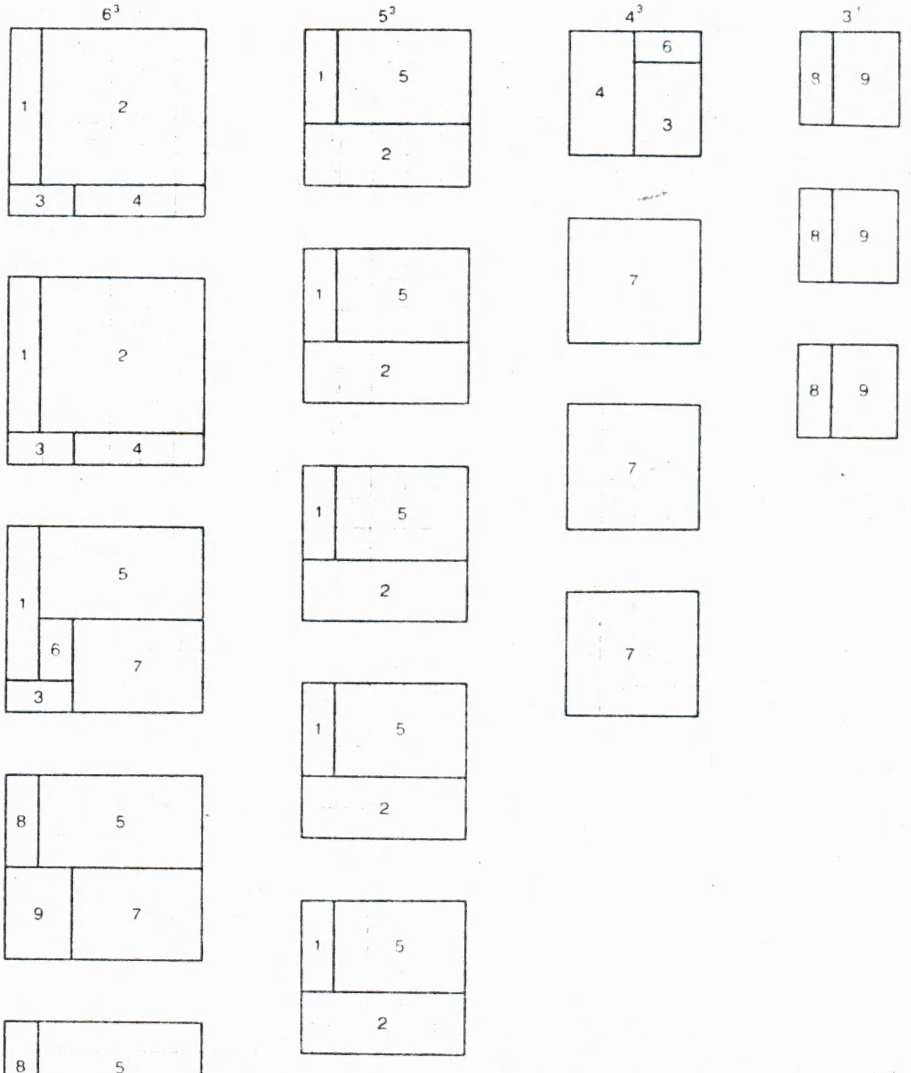
On August 13 *Newsweek* ran a story about Papert's turtle. A girl who had ranked near the bottom of her mathematics class was programming the turtle to draw a certain design. "That math must be fun," said a passing visitor. "There ain't *nothin'* fun in math," the girl replied. She had no idea she was doing math, reported *Newsweek*, and Papert saw no reason to tell her.

Last month's problem was to improve on a dissection of the 7-square into six pieces that form squares of sides 2, 3 and 6. A five-piece solution is shown in the top illustration at right. This dissection has the minimal cutting length of 16 units.

The bottom illustration at the right shows in six cross sections how Thomas H. O'Beirne sliced a 6-cube into nine rectangular blocks (the minimum) that can be reassembled to make separate cubes of sides 3, 4 and 5. No two "bricks" in this remarkable dissection are alike. O'Beirne has shown that a dissection in eight blocks is not possible, but it is not yet known whether or not the nine-piece dissection is unique.



Five-piece dissection for $7^2 = 6^2 + 3^2 + 2^2$



Nine-block dissection for $6^3 = 5^3 + 4^3 + 3^3$